The Logic of Clue/Cluedo

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Definitions

• Each card identifies one item
• Each item is unique
• Each card is in exactly one location
• Each location may hold multiple cards
• There are three types of items: suspect, weapon, and room
• A location is either a player’s hand or the cellar
• The cellar location holds exactly one card of each type
• The goal of the game is to identify the three cards in the cellar
The “Logic Physics” of the Game

- Each card must be in exactly one location
- The cellar contains exactly one card of each type
- Each player, \( P_i \), holds \( N_i \) cards
First Order Logic Representation

Dictionary

- **Cellar**: The cellar location.
- **$P_i$**: The $i^{th}$ player.
- **$N_i$**: The number of cards held by the $i^{th}$ player.
- **$L_j$**: The $j^{th}$ location, one of $P_i$ or **Cellar**.
- **$S_k$**: The $k^{th}$ card of type $S$, suspect.
- **$W_l$**: The $l^{th}$ card of type $W$, weapon.
- **$R_m$**: The $m^{th}$ card of type $R$, room.
- **$Holds(L, C)$**: The location $L$ holds the card $C$. 
First Order Logic Representation

- $\forall_{c,L_j} \text{Holds}(L_j, c) \iff \neg(\text{Holds}(L_0, c) \lor \text{Holds}(L_1, c) \lor \ldots \lor \text{Holds}(L_n, c))$
  where $L_j$ is not included in the right side, $c$ is a variable representing any $R_k$, $W_i$, or $R_m$.

- $\forall_{S_k} \text{Holds}(\text{Cellar}, S_k) \iff$
  $\neg(\text{Holds}(\text{Cellar}, S_0) \lor \text{Holds}(\text{Cellar}, S_1) \lor \ldots \lor \text{Holds}(\text{Cellar}, S_n))$
  $S_k$ is not included on the right side.

- $\forall_{W_i} \text{Holds}(\text{Cellar}, W_i) \iff$
  $\neg(\text{Holds}(\text{Cellar}, W_0) \lor \text{Holds}(\text{Cellar}, W_1) \lor \ldots \lor \text{Holds}(\text{Cellar}, W_n))$
  $W_i$ is not included on the right side.

- $\forall_{R_m} \text{Holds}(\text{Cellar}, R_m) \iff$
  $\neg(\text{Holds}(\text{Cellar}, R_0) \lor \text{Holds}(\text{Cellar}, R_1) \lor \ldots \lor \text{Holds}(\text{Cellar}, R_n))$
  $R_m$ is not included on the right side.
Propositional Logic Representation

- $L_0$: The cellar location.
- $P_i$: The $i^{th}$ player.
- $N_i$: The number of cards held by the $i^{th}$ player.
- $L_j$: The $j^{th}$ location, one of $P_i$ or Cellar.
- $hL_j C$: The location $L_j$ holds the card $C$.
- $S_k$: The $k^{th}$ suspect card.
- $W_l$: The $l^{th}$ weapon card.
- $R_m$: The $m^{th}$ room card.
Propositional Logic Representation

- **$hL_j C \iff \neg (hL_0 C \lor hL_1 C \lor ... \lor hL_n C)$**
  where $L_j$ is not included in the right side.

- **$hL_0 S_k \iff \neg (hL_0 S_0 \lor hL_0 S_0 \lor ... \lor hL_0 S_n)$**
  $S_k$ is not included on the right side.

- **$hL_0 W_l \iff \neg (hL_0 W_0 \lor hL_0 W_0 \lor ... \lor hL_0 W_n)$**
  $W_l$ is not included on the right side.

- **$hL_0 R_m \iff \neg (hL_0 R_0 \lor hL_0 R_0 \lor ... \lor hL_0 R_n)$**
  $R_m$ is not included on the right side.
Conjunctive Normal Form Representation

- At most one of each card:
  \[ \neg hL_i C \lor \neg hL_j C \]
  where all combinations of \( i, j \) are applied for each card, \( C \).

- At least one of each card:
  \[ hL_0 C \lor hL_1 C \lor \ldots \lor hL_n C \]
  applied for each card, \( C \).

- At most one card of each type in the cellar
  \[ \neg hL_0 T_i \lor \neg hL_0 T_j \]
  where all combinations of \( i, j \) are applied for each card type, \( T = S, W, R \).

- At least one card of each type in the cellar:
  \[ hL_0 T_0 \lor hL_0 T_1 \lor \ldots \lor hL_0 T_n \]
  applied for each card type, \( T = S, W, R \).
Definitions

- Our player has a fixed set of cards in hand.
- Players may make suggestions or accusations.
- A suggestion is processed clockwise around the board until disproved or all other players have not been able to disprove.
- A player may disprove a suggestion, and the processing stops.
- A player must disprove a suggestion, if possible.
- A player may not be able to disprove a suggestion, so processing continues.
- An accusation may be correct or incorrect.
The “Instance Logic” of the Game

- If we know all of our player’s cards, then we also know which ones we don’t hold.
- Not disproving a suggestion proves a player doesn’t hold any of the cards.
- Disproving a suggestion proves a player holds at least one of the cards.
- If no player disproves a suggestion, either the suggester or the cellar holds each card.
- If a player’s accusation fails, then at least one of the cards is not in the cellar.
Conjunctive Normal Form Representation

- Player $i$ holds card $C$: $hL_iC$
- Player $i$ doesn’t hold card $C$: $\neg hL_iC$
- Player $i$ doesn’t disprove $S_k, W_l, R_m$:
  $$(\neg hL_iS_k) \land (\neg hL_iW_l) \land (\neg hL_iR_m)$$
- Player $i$ disproves $S_k, W_l, R_m$: $hL_iS_k \lor hL_iW_l \lor hL_iR_m$
- No player disproves $S_k, W_l, R_m$, suggested by player $i$:
  $$(hL_iS_k \lor hL_0S_k) \land (hL_iW_l \lor hL_0W_l) \land (hL_iR_m \lor hL_0R_m)$$
- Accusation $S_k, W_l, R_m$ is false:
  $$(\neg hL_0S_k) \lor (\neg hL_0W_l) \lor (\neg hL_0R_m)$$
The “Questions” of the Game

• We want to know which card of each type is held by the cellar.
Conjunctive Normal Form Representation

- Cellar holds suspect card \( S_k: hL_0S_k \)
- Cellar holds weapon card \( W_i: hL_0W_i \)
- Cellar holds room card \( R_m: hL_0R_m \)